LITTLEWOOD POLYNOMIALS WITH SMALL L^4 NORM

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ABSTRACT

Littlewood asked how small the ratio $||f||_4/||f||_2$ (where $||\cdot||_\alpha$ denotes the L^α norm on the unit circle) can be for polynomials f having all coefficients in $\{1, -1\}$, as the degree tends to infinity. Since 1988, the least known asymptotic value of this ratio has been $\sqrt[4]{7/6}$, which was conjectured to be minimum. We disprove this conjecture by showing that there is a sequence of such polynomials, derived from the Fekete polynomials, for which the limit of this ratio is less than $\sqrt[4]{22/19}$.

1. INTRODUCTION

The L^α norm on the unit circle of polynomials having all coefficients in $\{1, -1\}$ (*Littlewood polynomials*) has attracted sustained interest over the last sixty years [36], [13], [33], [30], [26], [27], [34], [31], [2], [10]. For $1 \le \alpha < \infty$, the L^α norm of a polynomial $f \in \mathbb{C}[z]$ on the unit circle is

$$||f||_\alpha = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^\alpha \, d\theta \right)^{1/\alpha},$$

while $||f||_\infty$ is the supremum of $|f(z)|$ on the unit circle. The norms L^1, L^2, L^4, and L^∞ are of particular interest in analysis.

Littlewood was interested in how closely the ratio $||f||_\infty/||f||_2$ can approach 1 as $\deg(f) \to \infty$ for f in the set of polynomials now named after him [26]. Note that if f is a Littlewood polynomial, then $||f||_2^2$ is $\deg(f) + 1$. In view of the monotonicity of L^α norms, Littlewood and subsequent researchers used $||f||_4/||f||_2$ as a lower bound for $||f||_\infty/||f||_2$. The L^4 norm is particularly suited to this purpose because it is easier to calculate than most other L^α norms. The L^4 norm is also of interest in the theory of communications, because $||f||_4^4$ equals the sum of squares of the aperiodic autocorrelations of the sequence formed from the coefficients of f [20, eqn. (4.1)],

[4, p. 122]; in this context one considers the *merit factor* $||f||_2^4/(||f||_4^4-||f||_2^4)$. We shall express merit factor results in terms of $||f||_4/||f||_2$.

If $||f||_4/||f||_2$ is bounded away from 1, then so is $||f||_\infty/||f||_2$, which would prove a modification of a conjecture due to Erdős [14], [13, Problem 22] asserting that there is some $c > 0$ such that $||f||_\infty/||f||_2 \geq 1+c$ for all non-constant polynomials f whose coefficients have absolute value 1. It is known from Kahane's work that there is no such c [24], but the modified conjecture where f is restricted to be a Littlewood polynomial remains resistant [31].

Littlewood regarded calculations carried out by Swinnerton-Dyer as evidence that the ratio $||f||_4/||f||_2$ can be made arbitrarily close to 1 for Littlewood polynomials [26]. However, he could prove nothing stronger than that this ratio is asymptotically $\sqrt[4]{4/3}$ for the Rudin-Shapiro polynomials [27, Chapter III, Problem 19]. Høholdt and Jensen, building on studies due to Turyn and Golay [18], proved in 1988 that this ratio is asymptotically $\sqrt[4]{7/6}$ for a sequence of Littlewood polynomials derived from the Fekete polynomials [20]. Høholdt and Jensen conjectured that no further reduction in the asymptotic value of $||f||_4/||f||_2$ is possible for Littlewood polynomials. Although Golay conjectured, based on heuristic reasoning, that the minimum asymptotic ratio $||f||_4/||f||_2$ for Littlewood polynomials is approximately $\sqrt[4]{333/308}$ [17], he later cautioned that "the eventuality must be considered that no systematic synthesis will ever be found which will yield [a smaller asymptotic ratio than $\sqrt[4]{7/6}$]" [18].

Indeed, for more than twenty years $\sqrt[4]{7/6}$ has remained the smallest known asymptotic value of $||f||_4/||f||_2$ for a sequence of Littlewood polynomials f. We shall prove that this is not the minimum asymptotic value.

Theorem 1.1. *There is a sequence $h_1, h_2, \ldots$ of Littlewood polynomials with $\deg(h_n) \to \infty$ and $||h_n||_4/||h_n||_2 \to \sqrt[4]{c}$ as $n \to \infty$, where $c < 22/19$ is the smallest root of $27x^3 - 498x^2 + 1164x - 722$.*

To date, two principal methods have been used to calculate the L^4 norm of a sequence of polynomials [19]. The first is direct calculation, in the case that the polynomials are recursively defined [27]. The second, introduced by Høholdt and Jensen [20] and subsequently employed widely for its generality [22], [23], [5], [6], [4], [7], [35], [21], obtains $||f||_4$ from a Fourier interpolation of f. In this paper, we use a simpler method that also obtains the L^4 norm of truncations and periodic extensions of f. We apply this method to Littlewood polynomials derived from the Fekete polynomials, themselves the fascinating subject of many studies [15], [32], [28], [1], [11], [9], [7]. The possibility that these derived polynomials could have an asymptotic ratio $||f||_4/||f||_2$ smaller than $\sqrt[4]{7/6}$ was first recognized by Kirilusha and Narayanaswamy [25] in 1999. Borwein, Choi, and Jedwab subsequently used extensive numerical data to conjecture conditions under which the value $\sqrt[4]{c}$ in Theorem 1.1 could be attained asymptotically (giving a corresponding asymptotic merit factor $1/(c - 1) > 6.34$) [8], but until now no explanation

has been given as to whether their conjecture might be correct, nor if so why.

2. The Asymptotic L^4 Norm of Generalized Fekete Polynomials

Henceforth, let p be an odd prime and let r and t be integers with $t \geq 0$. The *Fekete polynomial* of degree $p - 1$ is $\sum_{j=0}^{p-1} (j \,|\, p) z^j$, where $(\cdot \,|\, p)$ is the Legendre symbol. We define the *generalized Fekete polynomial*

$$f_p^{(r,t)}(z) = \sum_{j=0}^{t-1} (j + r \,|\, p) z^j.$$

The polynomial $f_p^{(r,t)}$ is formed from the Fekete polynomial of degree $p-1$ by cyclically permuting the coefficients through r positions, and then truncating when $t < p$ or periodically extending when $t > p$. We wish to determine the asymptotic behavior of the L^4 norm of the generalized Fekete polynomials for all r, t.

Since the Legendre symbol is a multiplicative character, we can use

$$||f||_4^4 = \frac{1}{2\pi} \int_0^{2\pi} \left[f(e^{i\theta}) \overline{f(e^{i\theta})} \right]^2 d\theta$$

to obtain

$$||f_p^{(r,t)}||_4^4 = \sum_{\substack{0 \leq j_1, j_2, j_3, j_4 < t \\ j_1 + j_2 = j_3 + j_4}} ((j_1 + r)(j_2 + r)(j_3 + r)(j_4 + r) \,|\, p). \tag{1}$$

Until now, the asymptotic evaluation of (1) has been considered intractable because, when t is not a multiple of p, the expression (1) is an incomplete character sum whose indices are subject to an additional constraint. We shall overcome this apparent difficulty by using the Fourier expansion of the multiplicative character $(j \,|\, p)$ in terms of additive characters of $\mathbb{F}_p$, with Gauss sums playing the part of Fourier coefficients. This expansion introduces complete character sums over $\mathbb{F}_p$ which, once computed, allow an easy asymptotic evaluation of (1). This method is considerably simpler and more general than the Fourier interpolation method of [20].

Theorem 2.1. *Let $r/p \to R < \infty$ and $t/p \to T < \infty$ as $p \to \infty$. Then*

$$\frac{||f_p^{(r,t)}||_4^4}{p^2} \to -\frac{4T^3}{3} + 2 \sum_{n \in \mathbb{Z}} \max(0, T - |n|)^2 + \sum_{n \in \mathbb{Z}} \max(0, T - |T + 2R - n|)^2$$

as $p \to \infty$.

Proof. Let $\epsilon_j = e^{2\pi i j/p}$ for $j \in \mathbb{Z}$. Gauss [16], [3] showed that

$$\sum_{k \in \mathbb{F}_p} \epsilon_j^k (k \,|\, p) = i^{(p-1)^2/4} \sqrt{p} \, (j \,|\, p).$$

Substitution in (1) gives

$$||f_p^{(r,t)}||_4^4 = \frac{1}{p^2} \sum_{\substack{0 \le j_1,j_2,j_3,j_4 < t \\ j_1+j_2=j_3+j_4}} \sum_{k_1,k_2,k_3,k_4 \in \mathbb{F}_p} \epsilon_{j_1+r}^{k_1} \epsilon_{j_2+r}^{k_2} \epsilon_{j_3+r}^{k_3} \epsilon_{j_4+r}^{k_4} (k_1 k_2 k_3 k_4 \,|\, p).$$

Re-index with $k_1 = x$, $k_2 = x - a$, $k_3 = b - x$, $k_4 = c - x$ to obtain

$$||f_p^{(r,t)}||_4^4 = \frac{1}{p^2} \sum_{\substack{0 \le j_1,j_2,j_3,j_4 < t \\ j_1+j_2=j_3+j_4}} \sum_{a,b,c \in \mathbb{F}_p} \epsilon_a^{-(j_2+r)} \epsilon_b^{j_3+r} \epsilon_c^{j_4+r} L(a,b,c), \qquad (2)$$

where

$$L(a,b,c) = \sum_{x \in \mathbb{F}_p} (x(x-a)(x-b)(x-c) \,|\, p).$$

A Weil-type bound on character sums [37], [29, Lemma 9.25], shows that $|L(a,b,c)| \le 3\sqrt{p}$ when $x(x-a)(x-b)(x-c)$ is not a square in $\mathbb{F}_p[x]$. This polynomial is a square in $\mathbb{F}_p[x]$ if and only if it either has two distinct double roots, in which case $L(a,b,c) = p-2$, or else has a quadruple root, in which case $L(a,b,c) = p - 1$. We shall see that contributions from $L(a,b,c)$ much smaller than p will not influence the asymptotic value of the L^4 norm. Accordingly, we write $L(a,b,c) = M(a,b,c) + N(a,b,c)$, with a main term

$$M(a,b,c) = \begin{cases} p & \text{if } x(x-a)(x-b)(x-c) \text{ is a square in } \mathbb{F}_p[x], \\ 0 & \text{if } x(x-a)(x-b)(x-c) \text{ is not a square in } \mathbb{F}_p[x], \end{cases}$$

and an error term $N(a,b,c)$ satisfying

$$|N(a,b,c)| \le 3\sqrt{p}. \qquad (3)$$

There are three ways of pairing the roots of $x(x-a)(x-b)(x-c)$: (i) $a = c$ and $b = 0$, (ii) $b = a$ and $c = 0$, or (iii) $c = b$ and $a = 0$. So $M(a,b,c) = p$ if at least one of these conditions is met, and vanishes otherwise. The only triple (a,b,c) that satisfies more than one of these conditions is $(0,0,0)$. We now reorganize (2) by writing $L(a,b,c)$ as $M(a,b,c) + N(a,b,c)$, and then break the sum involving $M(a,b,c)$ into four parts: three sums corresponding to the three pairings, and a fourth sum to correct for the triple counting of $(a,b,c) = (0,0,0)$. We keep the sum over $N(a,b,c)$ entire, and thus have

$$||f_p^{(r,t)}||_4^4 = A + B + C + D + E, \qquad (4)$$

where

$$
\begin{aligned}
A &= \frac{1}{p} \sum_{\substack{0 \le j_1,j_2,j_3,j_4 < t \\ j_1+j_2=j_3+j_4}} \sum_{a \in \mathbb{F}_p} \epsilon_a^{j_4-j_2}, \\[4pt]
B &= \frac{1}{p} \sum_{\substack{0 \le j_1,j_2,j_3,j_4 < t \\ j_1+j_2=j_3+j_4}} \sum_{b \in \mathbb{F}_p} \epsilon_b^{j_3-j_2}, \\[4pt]
C &= \frac{1}{p} \sum_{\substack{0 \le j_1,j_2,j_3,j_4 < t \\ j_1+j_2=j_3+j_4}} \sum_{c \in \mathbb{F}_p} \epsilon_c^{j_3+j_4+2r}, \\[4pt]
D &= -\frac{2}{p} \sum_{\substack{0 \le j_1,j_2,j_3,j_4 < t \\ j_1+j_2=j_3+j_4}} 1, \\[4pt]
E &= \frac{1}{p^2} \sum_{a,b,c \in \mathbb{F}_p} N(a,b,c)\epsilon_{-a+b+c}^r \sum_{\substack{0 \le j_1,j_2,j_3,j_4 < t \\ j_1+j_2=j_3+j_4}} \epsilon_a^{-j_2}\epsilon_b^{j_3}\epsilon_c^{j_4}.
\end{aligned}
$$

Note that $A = B$, and that A counts the quadruples (j_1, j_2, j_3, j_4) of integers in $[0, t)$ with $j_1 + j_2 = j_3 + j_4$ and $j_4 \equiv j_2 \pmod{p}$, or equivalently, with $j_4 - j_2 = np$ and $j_3 - j_1 = -np$ for some $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$ there are $\max(0, t - |n|p)$ ways to obtain $j_4 - j_2 = np$ and the same number of ways to obtain $j_3 - j_1 = -np$. Therefore $A = B = \sum_{n \in \mathbb{Z}} \max(0, t - |n|p)^2$. This is a locally finite sum of continuous functions, and since $t/p \to T$ as $p \to \infty$ we have

$$
\frac{A}{p^2} + \frac{B}{p^2} \to 2 \sum_{n \in \mathbb{Z}} \max(0, T - |n|)^2.
$$

The summation in D counts the quadruples (j_1, j_2, j_3, j_4) of integers in $[0, t)$ with $j_1 + j_2 = j_3 + j_4$. For each $n \in \mathbb{Z}$ there are $\max(0, t - |t - 1 - n|)$ ways to represent n as $j_1 + j_2$ with $j_1, j_2 \in [0, t)$, and so

$$
D = -\frac{2}{p} \sum_{n \in \mathbb{Z}} \max(0, t - |t - 1 - n|)^2.
$$

Thus $D = -\frac{2t(2t^2+1)}{3p}$, and since $t/p \to T$ as $p \to \infty$ we get $D/p^2 \to -4T^3/3$.

Similarly, C counts the quadruples (j_1, j_2, j_3, j_4) of integers in $[0, t)$ with $j_1 + j_2 = j_3 + j_4 = -2r + np$ for some $n \in \mathbb{Z}$. Replacing n by $-2r + np$ in the above argument for D, we find that

$$
\frac{C}{p^2} = \sum_{n \in \mathbb{Z}} \max\left(0, \frac{t}{p} - \left|\frac{t-1+2r}{p} - n\right|\right)^2.
$$

This is a locally finite sum of continuous functions $\psi_n(x, y) = \max(0, x - |y - n|)^2$ evaluated at $x = t/p$ and $y = (t - 1 + 2r)/p$. Since $r/p \to R$ and

$t/p \to T$ as $p \to \infty$, it follows that

$$\frac{C}{p^2} \to \sum_{n \in \mathbb{Z}} \max(0, T - |T + 2R - n|)^2.$$

By (4), it remains to show that $|E|/p^2 \to 0$ as $p \to \infty$. Use (3) to bound $|N(a,b,c)|$, and then use Lemma 2.2 below to bound the resulting outer sum over a, b, c to give $|E|/p^2 \le 192 p^{-7/2} \max(p,t)^3 (1 + \log p)^3$. Since $t/p \to T < \infty$ as $p \to \infty$, we then obtain $E/p^2 \to 0$ as required. $\qquad \square$

We now prove the technical result invoked in the proof of Theorem 2.1.

Lemma 2.2. *Let n be a positive integer and $\epsilon_j = e^{2\pi i j / n}$ for $j \in \mathbb{Z}$. Then*

$$\sum_{a,b,c \in \mathbb{Z}/n\mathbb{Z}} \left| \sum_{\substack{0 \le j_1, j_2, j_3, j_4 < t \\ j_1 + j_2 = j_3 + j_4}} \epsilon_a^{-j_2} \epsilon_b^{j_3} \epsilon_c^{j_4} \right| \le 64 \max(n, t)^3 (1 + \log n)^3.$$

Proof. Let G be the entire sum. Re-index with $k = -a$, $\ell = c - b$, $m = b$, to obtain

$$G = \sum_{k,\ell,m \in \mathbb{Z}/n\mathbb{Z}} \left| \sum_{\substack{0 \le j_1, j_2, j_3, j_4 < t \\ j_1 + j_2 = j_3 + j_4}} \epsilon_k^{j_2} \epsilon_\ell^{j_4} \epsilon_m^{j_3 + j_4} \right|.$$

Re-index the inner sum with $h = j_3 + j_4$, separating into ranges $h \le t - 1$ and $h \ge t$, so that $G \le H + J$, where

$$H = \sum_{k,\ell,m \in \mathbb{Z}/n\mathbb{Z}} \left| \sum_{h=0}^{t-1} \epsilon_m^h \sum_{j_2, j_4 = 0}^{h} \epsilon_k^{j_2} \epsilon_\ell^{j_4} \right|,$$

$$J = \sum_{k,\ell,m \in \mathbb{Z}/n\mathbb{Z}} \left| \sum_{h=t}^{2t-2} \epsilon_m^h \sum_{j_2, j_4 = h-(t-1)}^{t-1} \epsilon_k^{j_2} \epsilon_\ell^{j_4} \right|.$$

We shall show that $H \le 32 \max(n, t)^3 (1 + \log n)^3$, from which we can deduce the same bound on J after re-indexing with $h' = 2(t-1) - h$, $j_2' = j_2 + h' - (t-1)$, $j_4' = j_4 + h' - (t-1)$, and $m' = -(k + \ell + m)$.

Partition the sum H into a sum with $k, \ell \ne 0$, two sums where one of k, ℓ is zero and the other is nonzero, and a sum where $k = \ell = 0$; then sum over

the indices j_2 and j_4 to obtain $H = H_1 + 2H_2 + H_3$, where

$$H_1 = \sum_{\substack{k,\ell,m\in\mathbb{Z}/n\mathbb{Z} \\ k,\ell\neq 0}} \left| \sum_{h=0}^{t-1} \frac{\epsilon_m^h - \epsilon_k\epsilon_{m+k}^h - \epsilon_\ell\epsilon_{m+\ell}^h + \epsilon_{k+\ell}\epsilon_{m+k+\ell}^h}{(1-\epsilon_k)(1-\epsilon_\ell)} \right|,$$

$$H_2 = \sum_{\substack{k,m\in\mathbb{Z}/n\mathbb{Z} \\ k\neq 0}} \left| \sum_{h=0}^{t-1} \frac{(h+1)\left(\epsilon_m^h - \epsilon_k\epsilon_{m+k}^h\right)}{1-\epsilon_k} \right|, \tag{5}$$

$$H_3 = \sum_{m\in\mathbb{Z}/n\mathbb{Z}} \left| \sum_{h=0}^{t-1}(h+1)^2\epsilon_m^h \right|,$$

To bound these sums, we prove by induction on $d \geq 0$ that for $s \geq 0$,

$$\sum_{\substack{j\in\mathbb{Z}/n\mathbb{Z} \\ j\neq 0}} \left| \sum_{h=0}^{s-1}(h+1)^d\epsilon_j^h \right| \leq 2^{d+1}s^d n \log n. \tag{6}$$

The base case follows from $\left| \sum_{h=0}^{s-1}\epsilon_j^h \right| \leq 2/|1-\epsilon_j|$ and the bound [12, p. 136]

$$\sum_{j=1}^{n-1} \frac{1}{|1-\epsilon_j|} \leq n\log n. \tag{7}$$

For the inductive step, apply the triangle inequality to the identity

$$\sum_{h=0}^{s-1}(h+1)^d\epsilon_j^h = \sum_{g=0}^{s-1}\sum_{h=0}^{s-1}(h+1)^{d-1}\epsilon_j^h - \sum_{g=0}^{s-1}\sum_{h=0}^{g-1}(h+1)^{d-1}\epsilon_j^h,$$

to obtain a bound for the left hand side as the sum of the magnitudes of $2s$ summations over h involving $(h+1)^{d-1}\epsilon_j^h$, then sum over $j \neq 0$ and use the inductive hypothesis.

Now from (6) we find

$$\sum_{j\in\mathbb{Z}/n\mathbb{Z}} \left| \sum_{h=0}^{t-1}(h+1)^d\epsilon_j^h \right| \leq t^{d+1} + 2^{d+1}t^d n \log n, \tag{8}$$

since $\sum_{h=0}^{t-1}(h+1)^d \leq t^{d+1}$. Apply the triangle inequality, (7), and (8) to the expressions (5) to obtain $H_1 \leq 4(t + 2n\log n)(n\log n)^2$, $H_2 \leq 2(t^2 + 4tn\log n)n\log n$, and $H_3 \leq t^3 + 8t^2 n\log n$. Therefore $H = H_1 + 2H_2 + H_3 \leq 32\max(n,t)^3(1+\log n)^3$, as required. $\square$

3. LITTLEWOOD POLYNOMIALS WITH SMALL L^4 NORM

The generalized Fekete polynomial $f_p^{(r,t)}$ is not necessarily a Littlewood polynomial, because its coefficient of z^j is 0 for $0 \leq j < t$ and $j + r \equiv 0$

(mod p). Replace each such zero coefficient of $f_p^{(r,t)}$ with 1 to define a family of Littlewood polynomials

$$g_p^{(r,t)}(z) = f_p^{(r,t)}(z) \; + \sum_{\substack{0 \le j < t \\ j+r \equiv 0 \,(\text{mod } p)}} z^j. \tag{9}$$

We now show that the asymptotic L^4 norm of $g_p^{(r,t)}$ as $p \to \infty$ behaves in the same way as that of $f_p^{(r,t)}$.

Corollary 3.1. *Let $r/p \to R < \infty$ and $t/p \to T < \infty$ as $p \to \infty$. Then*

$$\frac{||g_p^{(r,t)}||_4^4}{p^2} \to -\frac{4T^3}{3} + 2\sum_{n \in \mathbb{Z}} \max(0, T - |n|)^2 + \sum_{n \in \mathbb{Z}} \max(0, T - |T + 2R - n|)^2.$$

as $p \to \infty$.

Proof. Write $f = f_p^{(r,t)}$ and $g = g_p^{(r,t)}$ and $v = \lceil t/p \rceil$. Since the L^4 norm of each z^j in (9) is 1, the triangle inequality for the L^4 norm gives

$$\frac{1}{p^2}\left| ||g||_4^4 - ||f||_4^4 \right| \le \frac{1}{p^2}\left(4v||f||_4^3 + 6v^2||f||_4^2 + 4v^3||f||_4 + v^4 \right).$$

The limit as $p \to \infty$ of the right hand side is 0, because $||f||_4/\sqrt{p}$ has a finite limit by Theorem 2.1 and because $v/\sqrt{p} \to 0$ follows from $t/p \to T < \infty$. $\quad\square$

The specialization of Corollary 3.1 to $T = 1$ and $|R| \le 1/2$ recovers the result due to Høholdt and Jensen [20] that the asymptotic ratio $||g_p^{(r,p)}||_4/||g_p^{(r,p)}||_2$ is $\sqrt[4]{7/6 + 8(|R| - 1/4)^2}$ (which achieves its minimum value of $\sqrt[4]{7/6}$ at $R = \pm 1/4$). The specialization of Corollary 3.1 to $T \in (0, 1]$ proves the conjecture of Borwein, Choi, and Jedwab [8, Conjecture 7.5] mentioned at the end of the Introduction. The authors of [8] gave a proof that, subject to the truth of their conjecture, Theorem 1.1 holds. In fact, Theorem 1.1 follows from Corollary 3.1 directly. We now show this, and demonstrate that the asymptotic ratio $||g_p^{(r,t)}||_4/||g_p^{(r,t)}||_2$ for $T > 0$ and arbitrary R cannot be made less than the value $\sqrt[4]{c}$ given in Theorem 1.1.

Corollary 3.2. *If $r/p \to R < \infty$ and $t/p \to T \in (0, \infty)$ as $p \to \infty$ then*

$$\lim_{p \to \infty} \frac{||g_p^{(r,t)}||_4}{||g_p^{(r,t)}||_2} \ge \sqrt[4]{c},$$

where $c < 22/19$ is the smallest root of $27x^3 - 498x^2 + 1164x - 722$, with equality if and only if T is the middle root T_0 of $4x^3 - 30x + 27$ and $R = \frac{1}{4}(3 - 2T_0) + \frac{n}{2}$ for some integer n. If $t/p \to \infty$ as $p \to \infty$, then $||g_p^{(r,t)}||_4/||g_p^{(r,t)}||_2 \to \infty$ as $p \to \infty$.

Proof. Recall that $||f||_2^2 = t$ for a Littlewood polynomial f of degree $t-1$. In the case $t/p \to \infty$, the required result is an easy consequence of Lemma 3.3 below. This leaves the case where $r/p \to R < \infty$ and $t/p \to T \in (0, \infty)$ as $p \to \infty$. We have already noted that when $R = 1/4$ and $T = 1$, the

asymptotic ratio $||g_p^{(r,t)}||_4^4/||g_p^{(r,t)}||_2^4$ is $7/6$. If $t/p > 3/2$, we know from Lemma 3.3 that $||g_p^{(r,t)}||_4^4/||g_p^{(r,t)}||_2^4 \geq 1 + 2(1 - p/t)^2 > 11/9 > 7/6$, and so we may assume $T \leq 3/2$.

By Corollary 3.1, $\lim_{p\to\infty} ||g_p^{(r,t)}||_4^4/||g_p^{(r,t)}||_2^4$ is

$$\frac{1}{T^2}\left[-\frac{4T^3}{3} + 2\sum_{n\in\mathbb{Z}}\max(0, T - |n|)^2 + \sum_{n\in\mathbb{Z}}\max(0, T - |T + 2R - n|)^2\right].$$

Call this function $u(R, T)$ and note that it is always at least $-\frac{4T}{3} + 2$, so that $u(R, T) > 4/3$ if $T < 1/2$. By combination with the previous bound on T, we may assume $T \in [1/2, 3/2]$. Furthermore, $u(R, T)$ does not change when R is replaced by $R + 1/2$, so it is sufficient to consider points (R, T) in the set $D = [0, 1/2] \times [1/2, 3/2]$. We cover D with six compact sets:

$$\begin{aligned}
D_1 &= \{(R, T) \in D : T + 2R \leq 1\}, \\
D_2 &= \{(R, T) \in D : 1 \leq T + 2R, \ T + R \leq 1\}, \\
D_3 &= \{(R, T) \in D : 1 \leq T + R, \ T \leq 1\}, \\
D_4 &= \{(R, T) \in D : 1 \leq T, \ T + R \leq 3/2\}, \\
D_5 &= \{(R, T) \in D : 3/2 \leq T + R, \ T + 2R \leq 2\}, \\
D_6 &= \{(R, T) \in D : 2 \leq T + 2R\}.
\end{aligned}$$

These sets are chosen so that the restriction of $u(R, T)$ to D_k is a continuous rational function $u_k(R, T)$, and so $u(R, T)$ attains a minimum value on each D_k. For example,

$$u_4(R, T) = -\frac{4T}{3} + 2 + 4\frac{(T-1)^2}{T^2} + \frac{(1 - 2R)^2}{T^2} + \frac{(2T + 2R - 2)^2}{T^2}.$$

For each T, the function $u_4(R, T)$ is minimized when $R = (3 - 2T)/4$, and $u_4(\frac{1}{4}(3 - 2T), T) = \frac{1}{6T^2}(-8T^3 + 48T^2 - 60T + 27)$ is minimized on D_4 when T is the middle root T_0 of $4x^3 - 30x + 27$. Let $R_0 = (3 - 2T_0)/4$. The point (R_0, T_0) lies in the interior of D_4. One can show that $u_4(R_0, T_0)$ is the smallest root c of $27x^3 - 498x^2 + 1164x - 722$, and that $c < 22/19$.

Following the same method, the minimum value of $u_3(R, T)$ on D_3 is $7/6$, attained at $(1/4, 1)$. Partial differentiation with respect to R shows that the minimum of $u_2(R, T)$ on D_2 lies on the boundary with D_3, and that the minimum of $u_5(R, T)$ on D_5 lies on the boundary with D_4. The involution $(R, T) \mapsto (1 - R - T, T)$ maps D_1 onto D_2 while preserving the value of $u(R, T)$; likewise with $(R, T) \mapsto (2 - R - T, T)$ for D_6 and D_5. Therefore the unique global minimum of $u(R, T)$ on D is c, attained at (R_0, T_0). $\qquad\square$

We close by proving the bound on $||f||_4^4$ used in the proof of Corollary 3.2.

Lemma 3.3. *Let m be a positive integer, and let $f(z) = \sum_{j=0}^{t-1} f_j z^j$ be a Littlewood polynomial for which $f_j = f_k$ whenever $j \equiv k \pmod{m}$. Then*

$$||f||_4^4 \geq \sum_{n\in\mathbb{Z}}\max(0, t - |n|m)^2.$$

Proof. Note that $\overline{f(z)} = f(z^{-1})$ for z on the unit circle. By treating $f(z)$ and $f(z^{-1})$ as formal elements of $\mathbb{C}[z, z^{-1}]$, it is straightforward to show that $||f||_4^4$ is the sum of the squares of the coefficients of $f(z)f(z^{-1})$. For each $n \in \mathbb{Z}$, the coefficient of z^{nm} in $f(z)f(z^{-1})$ is

$$\sum_{\substack{0 \le j,k < t \\ j-k=nm}} f_j f_k.$$

By the periodicity of the coefficients of f, this equals the number of pairs of integers (j, k) in $[0, t)$ with $j - k = nm$, which is $\max(0, t - |n|m)$. Sum the square of this over $n \in \mathbb{Z}$ to obtain the desired bound. $\qquad\square$

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